Exam on Rewriting Theory (2 hours)

Frédéric Blanqui (INRIA)

The exam is made of 5 exercises that can be done in any order.
Answers must be written in French or in English.
All answers must be clearly justified with enough details.

Exercise 1 (4 points) Is the following system confluent and terminating?

1. \( \text{add}(\text{zero}, y) \rightarrow y \)
2. \( \text{add}(\text{succ}(x), y) \rightarrow \text{succ}(\text{add}(x, y)) \)
3. \( \text{mul}(\text{zero}, y) \rightarrow \text{zero} \)
4. \( \text{mul}(\text{succ}(x), y) \rightarrow \text{add}(y, \text{mul}(x, y)) \)
5. \( \text{add}(\text{add}(x, y), z) \rightarrow \text{add}(y, \text{add}(x, z)) \)
6. \( \text{mul}(\text{add}(x, y), z) \rightarrow \text{add}(\text{mul}(x, z), \text{mul}(y, z)) \)

Exercise 2 (2 points) Let \( R \) be a set of rewrite rules of the form \( f\vec{l} \rightarrow r \).
Let \( P \) be the set of (unmarked) dependency pairs \((l, r)\) of \( R \) such that \( r \) is not a strict subterm of \( l \). Prove that \( \rightarrow_R \) terminates if and only if \( \rightarrow_P \) terminates.

Exercise 3 (2 points) Let \( k \in \mathbb{Z} \). Prove that a rewrite system \( R \) terminates by using a monotone polynomial interpretation on \( D_k = \{ n \in \mathbb{Z} \mid n \geq k \} \) if and only if it terminates by using a monotone polynomial interpretation on \( D_0 \).

Exercise 4 (2 points) Let \( R \) be a relation on some set \( A \), and \( \leftrightarrow \) be a symmetric relation on \( A \). Let \( E = \leftrightarrow^* \) and \( \overline{R} = R^{-1} \), that is, \( tRt' \) iff \( uRu' \).
We say that:

- \( R \) is confluent modulo \( E \) iff \( \overline{R}E \overline{R} \subseteq R^*E \overline{R}^* \), that is, for all \( t, u, t', u' \) such that \( t\overline{R}t' \) \( E \) \( u \) \( \overline{R}u' \), there are \( v, w \) such that \( t\overline{R}v \) \( E \) \( w \overline{R}u' \).
- \( R \) is \( \leftrightarrow \)-locally confluent modulo \( E \) iff \( \overline{R}(R \cup \leftrightarrow) \subseteq R^*E \overline{R}^* \).

Prove that \( R \) is confluent modulo \( E \) if \( R \) is \( \leftrightarrow \)-locally confluent modulo \( E \) and \( \overline{R}E \overline{R} \) terminates.

Hint: since \( \overline{R}E \overline{R} \) terminates, every element has a normal form wrt \( R \).

Exercise 5 (2 points) An equation is closed if it contains no variable. Given a set \( E \) of equations, let \( =_E \) be the smallest monotone and stable equivalence relation containing \( E \). Prove that, for all finite sets of closed equations \( E \), \( =_E \) is decidable.

Hint: \( >_{\text{lpo}} \) is total on closed terms if \( > \) is total on function symbols.