Exam on Rewriting Theory (2 hours)

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The exam is made of 5 exercises that can be done in any order. Answers must be written in French or in English. All answers must be clearly justified with enough details.

Exercise 1 (4 points) Is the following system confluent and terminating?

1. add(zero, y) → y
2. add(succ(x), y) → succ(add(x, y))
3. mul(zero, y) → zero
4. mul(succ(x), y) → add(y, mul(x, y))
5. add(add(x, y, z) → add(y, mul(x, z), mul(y, z))

Solution. This system terminates because it is included in LPO with mul > add > succ, zero and mul, add ∈ lex.

1. add(zero, y) > y by LPO1 (subterm case).
2. add(succ(x), y) > succ(add(x, y)) by LPO2 (precedence case) since add > succ and add(succ(x), y) > add(x, y) by LPO3 (recursive call case) since:
   - add(succ(x), y) > x by LPO1 twice,
   - add(succ(x), y) > y by LPO1,
   - succ(x) > x by LPO1.
3. mul(zero, y) > zero by LPO2 since mul > zero.
4. mul(succ(x), y) > add(y, mul(x, y)) by LPO2 since mul > add and:
   - mul(succ(x), y) > y by LPO1,
   - mul(succ(x), y) > mul(x, y) by LPO3 since:
     - mul(succ(x), y) > x by LPO1 twice,
     - mul(succ(x), y) > y by LPO1,
     - succ(x) > x by LPO1.
5. add(add(x, y, z) > add(x, add(y, z)) by LPO3 since:
   - add(add(x, y, z) > x by LPO1 twice,
   - add(add(x, y, z) > add(y, z) by LPO3 since
     - add(add(x, y, z) > y by LPO1 twice,
     - add(add(x, y, z) > z by LPO1,
     - add(x, y) > y by LPO1;
   - add(x, y) > x by LPO1.
6. \( \text{mul}(\text{add}(x, y), z) > \text{add}(\text{mul}(x, z), \text{mul}(y, z)) \) by LPO2 since \( \text{mul} > \text{add} \) and:

- \( \text{mul}(\text{add}(x, y), z) > \text{mul}(x, z) \) by LPO3 since:
  - \( \text{mul}(\text{add}(x, y), z) > x \) by LPO1 twice,
  - \( \text{mul}(\text{add}(x, y), z) > z \) by LPO1,
  - \( \text{add}(x, y) > x \) by LPO1;
- \( \text{mul}(\text{add}(x, y), z) > \text{mul}(y, z) \) by LPO3 since:
  - \( \text{mul}(\text{add}(x, y), z) > y \) by LPO1 twice,
  - \( \text{mul}(\text{add}(x, y), z) > z \) by LPO1,
  - \( \text{add}(x, y) > y \) by LPO1.

Since it terminates, confluence follows from local confluence (Newman’s lemma), which follows from the confluence of critical pairs:

- 1 superposes 5 at position 1:
  \[ t = \text{add}(\text{add}(\text{zero}, y), z) \xrightarrow{\xi} \text{add}(\text{zero}, \text{add}(y, z)) \rightarrow \text{add}(y, z) = u \]
  \[ t \xrightarrow{1} u \]

- 2 superposes 5 at position 1:
  \[ t = \text{add}(\text{add}(\text{succ}(x), y), z) \xrightarrow{\xi} \text{add}(\text{succ}(x), \text{add}(y, z)) \rightarrow \text{succ}(\text{add}(x, \text{add}(y, z))) = u \]
  \[ t \xrightarrow{1} \text{add}(\text{succ}(\text{add}(x, y)), z) \rightarrow \text{succ}(\text{add}(x, y), z) \rightarrow u \]

- 1 superposes 6 at position 1:
  \[ t = \text{mul}(\text{add}(\text{zero}, y), z) \xrightarrow{\xi} \text{add}(\text{add}(\text{zero}, z), \text{mul}(y, z)) \rightarrow \text{add}(\text{zero}, \text{mul}(y, z)) \rightarrow \text{mul}(y, z) = u \]
  \[ t \xrightarrow{1} u \]

- 2 superposes 6 at position 1:
  \[ t = \text{mul}(\text{add}(\text{succ}(x), y), z) \xrightarrow{\xi} \text{add}(\text{mul}(\text{succ}(x), z), \text{mul}(y, z)) \rightarrow \text{add}(\text{mul}(x, \text{mul}(z), \text{mul}(y, z))) = u \]
  \[ t \xrightarrow{1} \text{mul}(\text{succ}(\text{add}(x, y)), z) \rightarrow \text{add}(z, \text{mul}(\text{add}(x, y), z)) \rightarrow u \]

- 5 superposes 5 at position 1:
  \[ t = \text{add}(\text{add}(\text{add}(x_1, x_2), y), z) \xrightarrow{\xi} \text{add}(\text{add}(x_1, x_2), \text{add}(y, z)) \rightarrow \text{add}(\text{add}(x_1, \text{add}(x_2, \text{add}(y, z))) = u \]
  \[ t \xrightarrow{1} \text{add}(\text{add}(x_1, \text{add}(x_2, y), z)) \rightarrow u \]

- 5 superposes 6 at position 1:
  \[ t = \text{mul}(\text{add}(\text{add}(x_1, x_2), y), z) \xrightarrow{\xi} \text{add}(\text{mul}(\text{add}(x_1, x_2), z), \text{mul}(y, z)) \rightarrow \text{add}(\text{mul}(x_1, \text{mul}(x_2, z), \text{mul}(y, z))) = u \]
  \[ t \xrightarrow{1} \text{mul}(\text{add}(x_1, \text{add}(x_2, y), z)) \rightarrow \text{add}(\text{mul}(x_1, z), \text{mul}(\text{add}(x_2, y), z)) \xrightarrow{2} u \]
Exercise 2 (2 points) Let \( \mathcal{R} \) be a set of rewrite rules of the form \( f^T \rightarrow r \). Let \( \mathcal{P} \) be the set of (unmarked) dependency pairs \( (l, r) \) of \( \mathcal{R} \) such that \( r \) is not a strict subterm of \( l \). Prove that \( \rightarrow_{\mathcal{R}} \) terminates if and only if \( (\rightarrow_{\mathcal{R}}^*) \cap \mathcal{P} \) terminates.

Solution. The proof is about the same as the one given in class/lecture notes. First, \( (\rightarrow_{\mathcal{R}}^*) \cap \mathcal{P} \subseteq \rightarrow_{\mathcal{P}} \mathcal{P} \subseteq (\rightarrow_{\mathcal{R}} \cup \mathcal{P})^+ \) which terminates since \( \rightarrow_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}} \mathcal{P} \) (since \( \rightarrow_{\mathcal{R}} \) is monotone) and both \( \rightarrow_{\mathcal{R}} \) and \( \rightarrow_{\mathcal{R}} \mathcal{P} \) terminate.

We have seen that, if \( t \) is a non-terminating term with terminating subterms, then there are \( l \rightarrow r \in \text{DP}(\mathcal{R}) \) and \( \sigma \) such that \( t(\rightarrow_{\mathcal{R}}^*)^l \sigma \mathcal{P} \mathcal{P} + \sigma \) and \( r \sigma \) is a non-terminating term with terminating subterms. Therefore, \( r \) cannot be a subterm of \( l \). ■

Exercise 3 (2 points) Let \( k \in \mathbb{Z} \). Prove that a rewrite system \( \mathcal{R} \) terminates by using a monotone polynomial interpretation on \( D_k = \{ n \in \mathbb{Z} \mid n \geq k \} \) if and only if it terminates by using a monotone polynomial interpretation on \( D_0 \).

Solution. Given a monotone polynomial interpretation \( I \) on \( D_p \), \( p \in \mathbb{Z} \), let \( \rightarrow_{p,t} \) be the reduction ordering on terms such that \( t \rightarrow_{p,t} u \) if, for all valuation \( \xi : \mathcal{V} \rightarrow D_p \), \( \phi(I, \xi, t) > \phi(I, \xi, u) \), where \( \phi(I, \xi, t) \) is the interpretation of \( t \) in \( D_p \) using \( I \) and \( \xi \). We prove that, if \( \mathcal{R} \subseteq \rightarrow_{p,t} \) then, for all \( q \in \mathbb{Z} \), there is a monotone interpretation \( J \) on \( D_{p+q} \) such that \( \mathcal{R} \subseteq \rightarrow_{p+q,J} \). For all function symbols \( f \) of arity \( n \), let \( J_I(x) = I_I(x) + q \). If \( x \geq p + q \) then \( x + q \geq p \), \( I_I(x) + q \geq p \). Moreover, \( J_I \) is monotone in every argument since \( I_I \) is so. We now prove that, for all terms \( t \), \( \phi(J, \xi, t) = \phi(I, \xi, t) + q \), by induction on \( t \):

- Case \( t = x \). \( \phi(J, \xi, t) = \xi(x) \) and \( \phi(I, \xi, t) = \xi(x) \).
- Case \( t = f(t_1, \ldots, t_n) \). \( \phi(J, \xi, t) = J_f(\phi(J, \xi, t_1), \ldots, \phi(J, \xi, t_n)) = \phi(I, \xi, t) + q \). By induction hypothesis, \( \phi(J, \xi, t_i) = \phi(I, \xi, t_i) \). Hence, \( \phi(J, \xi, t) = \phi(I, \xi, t) + q \). So, \( \rightarrow_{p,t} \subseteq \rightarrow_{p+q,J} \). By taking \( p = k \) and \( q = -k \), we get the “if” part. By taking \( p = 0 \) and \( q = k \), we get the “only if” part. ■

Exercise 4 (2 points) Let \( R \) be a relation on some set \( A \), and \( \leftrightarrow \) be a symmetric relation on \( A \). Let \( E = \leftrightarrow^* \) and \( \overline{R} = R^{-1} \), that is, \( t \overline{R} u \) iff \( u R t \). We say that:

- \( R \) is confluent modulo \( E \) iff \( \overline{R} E R^* \subseteq R^* E \overline{R} \), that is, for all \( t, u, t', u' \) such that \( t \overline{R} u E R^* u' \), there are \( v, w \) such that \( t R^* v E w R^* u' \).
- \( R \) is \( \leftrightarrow \)-locally confluent modulo \( E \) iff \( \overline{R} (R \cup \leftrightarrow) \subseteq R^* E \overline{R} \).

Prove that \( R \) is confluent modulo \( E \) if \( R \) is \( \leftrightarrow \)-locally confluent modulo \( E \) and \( E \) terminates.

Hint: since \( E \) terminates, every element has a normal form wrt \( R \).
Solution. We prove that, for all \((t, u), t', u'\), if \(t' \mathrel{\rightarrow^*} t EuR^* u'\), then \(t' \mathrel{\rightarrow^*} E \mathrel{\rightarrow^*} u'\), by induction on the multiset \([t, u]\) ordered with \((ERE)_{mul}\) (1).

If \(t = t'\) and \(u = u'\), this is immediate. So, wlog, assume that \(t R t'\).

We then proceed by induction on the number of \(\leftrightarrow\) steps between \(t\) and \(u\) (2).

- Case \(t = u\). If \(u = u'\), this is immediate. So, assume that \(u Ru_1 t u'\). By \(\leftrightarrow\)-local confluence, there are \(v_1\) and \(v_2\) such that \(t_1 \mathrel{\rightarrow^*} v_1 Ev_2 R u_1\). Let \(t''\), \(v'_1\), \(v'_2\) and \(u''\) be any \(R\)-normal form of \(t'\), \(v_1\), \(v_2\) and \(u'\) respectively. By induction hypothesis (1) on \((t_1, t_1)\), \(t'' Ev'_1\). By induction hypothesis (1) on \((v_1, v_2)\), \(v'_1 Ev'_2\). And by induction hypothesis (1) on \((u_1, u_1)\), \(v'_2 E u''\).

- Case \(t \leftrightarrow s Eu\). By \(\leftrightarrow\)-local confluence, there are \(v_1\) and \(v_2\) such that \(t_1 \mathrel{\rightarrow^*} v_1 Ev_2 R s\). Let \(t''\), \(v'_1\), \(v'_2\) and \(u''\) be any \(R\)-normal form of \(t'\), \(v_1\), \(v_2\) and \(u'\) respectively. By induction hypothesis (1) on \((t_1, t_1)\), \(t'' Ev'_1\). By induction hypothesis (1) on \((v_1, v_2)\), \(v'_1 Ev'_2\). And by induction hypothesis (2) on \(s\), \(v'_2 E u''\).

Exercise 5 (2 points) An equation is closed if it contains no variable. Given a set \(E\) of equations, let \(=_{E}\) be the smallest monotone and stable equivalence relation containing \(E\). Prove that, for all finite sets of closed equations \(E\), \(=_{E}\) is decidable.

Hint: \(>_lpo\) is total on closed terms if \(>\) is total on function symbols.

Solution. Let \(>\) be an arbitrary well-order on function symbols. Then, \(> = >_{lpo}\) is a monotone well-order on closed terms. We now apply Knuth-Bendix completion on \(E\) using \(>\). By (orient) and (delete), we can transform \(E\) into a set of rules \(R_0\) included in \(>\). Let now the rule (collapse-orient-delete) be the rule that replaces \(R \cup \{g \rightarrow d, l[g] \rightarrow r\}\) by:

1. \(R \cup \{g \rightarrow d, l[d] \rightarrow r\}\) if \(l[d] > r\),
2. \(R \cup \{g \rightarrow d, r \rightarrow l[d]\}\) if \(r > l[d]\),
3. \(R \cup \{g \rightarrow d\}\) if \(l[d] = r\).

This rule preserves and reflects the equational theory since it is a composition of the rules (collapse), (orient) and (delete). We now prove that it cannot be applied indefinitely by using the following measure on \(R\). Let \(\|R\|\) be the multiset inductively defined as follows: \(\|\emptyset\| = 0, \|R \cup \{l \rightarrow r\}\| = \|R\| + [l, r]\). In every case, the size of \(R\) decreases wrt. \(>_{mul}\):

1. because \(l[g] > l[d]\) since \(g > d\) and \(>\) is monotone,
2. idem,
3. because \(l[g] \rightarrow r\) is removed.

So, in the end, we get a set \(R\) of rules that terminates (since it is included in \(>\)), has no critical pair (since the rule (collapse) is not applicable) and such that \(=_{R} = =_{E}\). Therefore, \(=_{E}\) is decidable.