Homework on rewriting theory

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evaluation criteria: correctness, presentation and precision

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two disjoint signatures (i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$), $\mathcal{V}$ be a set of variables disjoint from $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{R}_1$ be a set of rewrite rules on $\mathcal{F}_1$ such that $\to_1 = \to_{\mathcal{R}_1}$ terminates on $\mathcal{T}_1 = \mathcal{T}(\mathcal{F}_1, \mathcal{V})$, and $\mathcal{R}_2$ be a set of rewrite rules on $\mathcal{F}_2$ such that $\to_2 = \to_{\mathcal{R}_2}$ terminates on $\mathcal{T}_2 = \mathcal{T}(\mathcal{F}_2, \mathcal{V})$. Then, let $\to$ be the rewrite relation on $\mathcal{T} = \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ generated by $\mathcal{R}_1 \cup \mathcal{R}_2$. We are going to study sufficient conditions for the termination of $\to$.

To this end, we will use the following notions.

A (multi-holes) context is a term of $C = \mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ where $\Box$, the empty context, is a new constant of arity 0. If $C$ is a context and $p_1, \ldots, p_n$ are the positions of the occurrences of $\Box$ in $C$ from left to right, then $C[t_1, \ldots, t_n]$ denotes the term of $C$ obtained by replacing the $i$-th occurrence of $\Box$ by $t_i$ for every $i$ in $\{1, \ldots, n\}$.

A symbol is of color $k \in \{1, 2\}$ if it belongs to $\mathcal{F}_k$. A non-empty context non-reduced to a variable is of color $k$ if it belongs to $C_k = \mathcal{T}(\mathcal{F}_k \cup \{\Box\}, \mathcal{V})$. The opposite color of $k$, written $\overline{k}$, is 2 if $k = 1$, and 1 if $k = 2$.

Every element of $\mathcal{T}$ is of the form $C[t_1, \ldots, t_n]$ with $C$ a variable or a non-empty context of color $k$ and every $t_i$ headed by a symbol of color $\overline{k}$. $C$ is called the cap of $t$ and is denoted by $\cap(t)$. The terms $t_1, \ldots, t_n$ are called the aliens of $t$. Their multiset is denoted by $\aliens(t)$.

**Exercise 1 (2 points)** Let $\to_h$ be the restriction of $\to$ to homogeneous terms, that is, the relation such that $t \to_h u$ iff $t \to u$ and both $t$ and $u$ belong to $\mathcal{T}_1 \cup \mathcal{T}_2$. Prove that $\to_h$ terminates.

**Exercise 2 (3 points)** The rank of a term $t \in \mathcal{T}$, $\rk(t)$, is the maximum number of color layers in $t$: $\rk(t) = 1 + \sup_{a \in \aliens(t)} \rk(a)$. Prove that the rank cannot increase by reduction: if $t \to u$, then $\rk(t) \geq \rk(u)$.

- **Hint 1**: Look how evolve $\cap(t)$ and $\aliens(t)$ when $t \to u$.
- **Hint 2**: Proceed by induction on $\rk(t)$.

**Exercise 3 (4 points)** A rewrite rule $l \to r$ is collapsing if $r$ is a variable. Prove that $\to$ terminates if both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-collapsing.

- **Hint**: Look how $\cap(t)$ and $\aliens(t)$ evolve when $t \to u$, and devise a lexicographic combination of well-founded relations to prove the termination of every term $t \in \mathcal{T}$.

**Exercise 4 (4 points)** Given a term $t$, we define $S(t)$ to be the multiset made of $t$, the aliens of $t$, the aliens of the aliens of $t$, . . . : $S(t) = \sum_{i \geq 1} S_i(t)$ where $S_1(t) = [t]$ and, for all $i \geq 1$, $S_{i+1}(t) = \sum_{a \in \aliens(t)} S_i(a)$.
A rewrite rule \( l \rightarrow r \) is duplicating if some variable has more occurrences in \( r \) than it has in \( l \). Prove that \( \rightarrow \) terminates if both \( R_1 \) and \( R_2 \) are non-duplicating.

Hint: Look how \( rk(t) \) and \( S(t) \) evolve when \( t \rightarrow u \), and devise a lexicographic combination of well-founded relations to prove the termination of every term.

**Exercise 5 (3 points)** Assume that \( R_1 \) is non-collapsing and non-duplicating.

Let
\[
\|t\| = \begin{cases} 
0 & \text{if } t \in \mathcal{V} \\
\sum_{a \in \text{aliens}(t)} \|a\| & \text{if } \text{cap}(t) \in \mathcal{T}_1 \\
1 + \sup_{a \in \text{aliens}(t)} \|a\| & \text{if } \text{cap}(t) \in \mathcal{T}_2
\end{cases}
\]

Prove that \( \|t\| \geq \|u\| \) whenever \( t \rightarrow u \).

A reduction \( t \rightarrow u \) is destructive at level 1 if it is done in \( \text{cap}(t) \) and \( t \) and \( u \) have different colors. It is destructive at level 2 if it is a destructive reduction at level 1 in some alien of \( t \).

Observe that, if the reduction is destructive at level 1 or 2, then \( \|t\| > \|u\| \).

**Exercise 6 (4 points)** Prove that \( \rightarrow \) terminates if \( R_1 \) is non-collapsing and non-duplicating.