Homework on rewriting theory

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The homework can be written in English or in French.
All answers must be justified.

Let $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ be a signature and $\mathcal{V}$ be an infinite set of variables.
Let $\preceq$ be the subterm relation, and $\prec$ be its strict part. Let $\succeq$ and $\succ$ be their inverse relations respectively.
Given a quasi-order $\succeq$, let $\preceq$ be its inverse and $\succ = \succeq \setminus \preceq$ be its strict part.

Reminders:

• A relation is stable if it is closed by substitution.
• A relation is monotone if it is closed by context.
• A rewrite relation is a relation stable and monotone.
• An interpretation of $\mathcal{F}$ in some set $A$ is given by a function $f_I : A^n \to A$ for each $f \in \mathcal{F}_n$. Then, given a relation $R$ on $A$, $R_I$ is the relation on terms such that $tR_Iu$ iff, for all valuation $\xi : \mathcal{V} \to A$, $t\xi Ru\xi$, where $t\xi$ is the interpretation of $t$ in $A$ wrt the valuation $\xi$.

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**Exercise 1** Prove that every terminating rewrite strict-order $\succ$ that is total (for all $t \neq u$, either $t \succ u$ or $t \prec u$) contains $\succ$.

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In this homework, we will explore the use of quasi-orders containing $\succ$ for proving termination.

**Exercise 2** Let $\succeq_F$ be a quasi-order on $\mathcal{F}$ whose strict part $\succ_F$ terminates, and $\succ_{mpo}$ be its associated MPO. Prove that $\succ_{mpo}$ is a rewrite strict-order containing $\succ$.

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**Exercise 3** Let $> : A \to A$ be a strict-order on a set $A$, and $I$ be an interpretation of $\mathcal{F}$ in $A$, that is, a function $f_I : A^n \to A$ for every $f \in \mathcal{F}_n$. Prove that:

(a) $>_{f}$ is a rewrite strict-order if the functions $f_I$ are strictly monotone ($f_I \ldots x \ldots > f_I \ldots y \ldots$ whenever $x > y$).

(b) $\geq_I$ is a rewrite quasi-order if the functions $f_I$ are monotone ($f_I \ldots x \ldots \geq f_I \ldots y \ldots$ whenever $x \geq y$).

(c) $\geq_I$ contains $\succeq$ if the functions $f_I$ are extensive ($f_I \ldots x \ldots \geq x$).
(d) \( \succeq_I \) contains \( \succeq \) if the functions \( f_I \) are strictly monotone (\( f_I \ldots x\ldots > f_I \ldots y\ldots \) whenever \( x > y \)) and \( > \) is a well-order on \( A \) (every non-empty subset of \( A \) has a least element).

**Exercise 4** Find a rewrite strict-order containing \( \succ \) that does not terminate.

**Exercise 5** Prove that a stable strict-order \( \succ \) such that \( \succeq \subseteq \succeq \) cannot introduce new variables, that is, if \( t \succ u \) then \( \mathcal{V}(u) \subseteq \mathcal{V}(t) \).

Let \( \sqsubseteq \) be the smallest relation on terms such that:

- \( \text{(var)} \) for all variables \( x, x \sqsubseteq x \);
- \( \text{(mon)} \) if \( f \in \mathcal{F}_n, t_1 \sqsubseteq u_1, \ldots, t_m \sqsubseteq u_m, \) then \( ft_1 \ldots t_m \sqsubseteq fu_1 \ldots u_m \);
- \( \text{(sub)} \) if \( t \sqsubseteq u_k, \) then \( t \sqsubseteq fu_1 \ldots u_m \).

Let \( \sqsubset \) be the strict part of \( \sqsubseteq \).

**Exercise 6** Prove the following properties:

(a) \( \sqsubseteq \) is reflexive.
(b) \( \sqsubseteq \) is transitive.
(c) \( \sqsubseteq \) is antisymmetric.
(d) \( \sqsubseteq \) is monotone.
(e) \( \sqsubseteq \) is stable.
(f) \( \sqsubseteq \) contains the subterm relation \( \sqsubseteq_t \).
(g) \( \sqsubseteq \) is the smallest rewrite order containing \( \sqsubseteq_t \).
(h) \( \sqsubseteq \) is the smallest rewrite strict-order containing \( \sqsubset_t \).
(i) \( \sqsubseteq \) is decidable if \( \mathcal{F} \) and \( \mathcal{V} \) are decidable.

(Hint: define \( F : \mathcal{T} \times \mathcal{T} \rightarrow \{0, 1\} \) and prove that \( F(t, u) = 1 \) iff \( t \sqsubseteq u \).)

**Exercise 7** Prove that the rewrite rule

\[
\text{div} \ (s \ x) \ (s \ y) \ \rightarrow \ s(\text{div} \ (\text{minus} \ x \ y) \ (s \ y))
\]

is included in no rewrite strict-order containing \( \succ \).
**Exercise 8** Prove that a rewrite strict-order containing $\triangleright$ terminates if $\mathcal{F}$ is finite.

Hint: use Kruskal theorem saying that in any infinite sequence of terms $(t_i)_{i \in \mathbb{N}}$ with a finite number of symbols (function or variable), there are $j < k$ such that $t_j \sqsubseteq t_k$.

**Exercise 9** Let $\mathcal{R}$ be a set of rewrite rules. Prove that $\rightarrow_{\mathcal{R}}$ terminates if $\mathcal{F}$ is finite and there is an interpretation $I$ of $\mathcal{F}$ in $\mathbb{R}$ such that $\mathcal{R} \subseteq >_I$ and, for every $f$, (a) $f_I \ldots x \ldots > f_I \ldots y \ldots$ whenever $x > y$, and (b) $f_I \ldots x \ldots > x$, where $>$ is the (non-terminating!) standard order on $\mathbb{R}$. 