1 Basic definitions and results

1.1 Tarski’s fixpoint theorem

We start by recalling a useful result for justifying inductive definitions due to Tarski [11]:

Lemma 1 If \((A, \leq)\) is a complete lattice (i.e. every subset \(X\) of \(A\) has a greatest lower bound \(\text{glb}(X)\) and a lowest upper bound \(\text{lub}(X)\)) and \(f : A \to A\) is monotone, then \(f\) has a least fixpoint \(\text{lfp}(f)\) (\(x\) is a fixpoint of \(f\) if \(f(x) = x\)).

Proof. Let \(X = \{x \in A | f(x) \leq x\}\) and \(a = \text{glb}(X)\). By definition, \(f(a) \leq a\). By monotony, \(f(f(a)) \leq f(a)\). Hence, \(f(a) \in X\). Therefore, \(a \leq f(a)\). ■

In particular, the powerset of a set \(A\), \(\wp(A)\), is a complete lattice wrt \(\subseteq\): given \(X \subseteq A\), \(\text{lub}(X) = \bigcup X\), \(\text{glb}(X) = \bigcap X\) if \(X \neq \emptyset\), and \(\text{glb}(\emptyset) = A\).

In fact, Tarski showed more than that: he showed that the set of all the fixpoints of \(f\) is a non-empty complete lattice. Later, Davis proved that this characterizes complete lattices: a poset where every monotone function has a fixpoint is complete [3].

1.2 Properties of relations

Given a relation \(R\), let:

- \(R(t) = \{u | tRu\}\);
- \(R^+\) be its transitive closure, that is, the smallest transitive relation containing \(R\);
- \(R^*\) be its reflexive and transitive closure;
- \(R^\square\) be its reflexive closure;
- \(R^{-1}\) be its inverse: \(tR^{-1}u\) iff \(uRt\).
• $R^l$ be the relation such that $tR^l u$ iff $tR^* u$ and $R(u) = \emptyset$.

An element $t$ is in normal form or irreducible wrt a relation $R$ if $R(t) = \emptyset$.

A relation $R$ is:

• entire if, for all $x$, there is $y$ such that $(x, y) \in R$.

• weakly normalizing if every element has at least one normal form.

• terminating (or strongly normalizing, or Noetherian) if there is no infinite sequence $t_0Rt_1R\ldots$.

If $R$ terminates, then $R^{-1}$ is well-founded (every non-empty subset has an element minimal wrt $R^{-1}$). But, in the context of rewriting theory, where $R$ is usually written $\rightarrow$, we often say that $R$ itself is well-founded (instead of speaking of the relation $\leftarrow$).

• a pre-order or quasi-order if it is reflexive and transitive (but not necessarily anti-symmetric).

Given a quasi-order $R$, let its strict part be $R - R^{-1}$ and its associated equivalence relation be its symmetric closure (smallest relation containing both $R$ and $R^{-1}$). Usually, if $R$ is denoted by $\leq$, then its strict part is denoted by $<$ and its associated equivalence relation by $\simeq$.

By abuse of language, we often say that a quasi-ordering terminates instead of saying that its strict part terminates.

Strong normalization implies weak normalization.

### 1.3 Well-founded induction

Well-founded induction is a very useful tool in many areas of computer science. It generalizes induction on natural numbers.

To justify it, we however need a restricted form of the Axiom of Choice:

**Definition 1 (Axiom of dependent choice)** If $A$ is non-empty and $R$ is an entire relation on $A$, then there exists an infinite $R$ sequence $t_0Rt_1R\ldots$, that is, a function $f : \mathbb{N} \to A$ such that, for all $k$, $f(k)Rf(k+1)$.

**Theorem 1** Let $R$ be a terminating relation on a set $A$, and $P$ be a subset of $A$. Then, $P = A$ if, for all $t \in A$, we have $t \in P$ whenever $R(t) \subseteq P$.

**Proof.** Assume that $P \neq A$. Let $R'$ be the restriction of $R$ to $A - P$. $R'$ terminates and, for all $t \in A - P$, there is $u \in R'(t)$. Hence, by the axiom of dependent choice, there is an infinite sequence of $R'$ steps. Contradiction. ■

**Example:** $R = \{(n+1, n) | n \in \mathbb{N}\}$ gives the usual induction principle on $\mathbb{N}$. 

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Lemma 2 (Newman, 1942, [9]) If $R$ is terminating and locally confluent, then $R$ is confluent.

Proof. By well-founded induction on $R$. Assume that $t R^* u_i$, $i \in \{1, 2\}$. If $t = u_1$ or $t = u_2$, then we are done. Otherwise, for all $i$, there is $t_i$ such that $t R t_i R^* u_i$. By local confluence, there is $u$ such that, for all $i$, $t_i R^* u$. Now, by induction hypothesis, there is $v$ such that $u R^* v$ and $u R^* v$. And since $t_2 R^* u R^* v$, by induction hypothesis again, there is $w$ such that $u_2 R^* w$ and $v R^* w$. $\blacksquare$

1.4 Product relations

Given two relations $R$ and $S$ on some respective sets $A$ and $B$, let:

- the parallel product relation of $R$ and $S$ be the relation $R \times_{par} S$ on $A \times B$ such that $(a, b) R \times_{par} S(a', b')$ if $a R a'$ and $b S b'$;

- the sequential product relation of $R$ and $S$ be the relation $R \times_{seq} S$ on $A \times B$ such that $(a, b) R \times_{seq} S(a', b')$ if either $a R a'$ and $b = b'$, or $a = a'$ and $b S b'$;

- the lexicographic product relation of $R$ and $S$ be the relation $R \times_{lex} S$ on $A \times B$ such that $(a, b) R \times_{lex} S(a', b')$ if either $a R a'$ or $a = a'$ and $b S b'$.

All these product relations preserve termination: if $R$ and $S$ terminate, then $R \times S$ terminate.

When $R$ and $S$ are quasi-orders, these constructions can be generalized by replacing $=$ by $\simeq$. 

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1.5 Finite sequences or words

Given a relation \( R \) on \( A \), the lexicographic extension of \( R \) on \( A^* \), \( R_{\text{lex}} \), is defined as follows: \( \vec{a} R_{\text{lex}} \vec{b} \) if there is \( i \leq \min\{|\vec{a}|, |\vec{b}|\} \) such that \( a_i R b_i \) and, for all \( j < i \), \( a_j = b_j \).

Note that \( R_{\text{lex}} \) does not generally terminate even though \( R \) so does: if \( R = \{(a, b)\} \), then \( a R_{\text{lex}} b \, ba R_{\text{lex}} bba R_{\text{lex}} \ldots \)

We can however define terminating restrictions of it. For instance, by bounding the number of letters that can be compared: given \( n \in \mathbb{N} \), let \( \vec{a} R_{n \text{lex}} \vec{b} \) if there is \( i \leq \min\{|\vec{a}|, |\vec{b}|, n\} \) such that \( a_i R b_i \) and, for all \( j < i \), \( a_j = b_j \).

1.6 Finite multisets

Finite multisets are a very useful termination tool. A finite multiset on a set \( A \) is a function \( M : A \to \mathbb{N} \) the support of which is finite (\( M(a) \) is called the multiplicity of \( a \) in \( M \)). Let \( \mathbb{M}(A) \) be the set of finite multisets on \( A \). A multiset can be seen as a set where elements can occur more than once. For more details about multisets, see for instance [5, 12, 1].

A relation \( > \) on \( A \) can be extended into a relation \( >_{\text{mul}} \) on \( \mathbb{M}(A) \) as follows: \( M >_{\text{mul}} N \) iff there are multisets \( X, Y \) and \( Z \) such that \( M = Z + X \), \( N = Z + Y \), \( X \neq \emptyset \) and \( Y \subseteq > (X) \), that is, for all \( y \in Y \), there is \( x \in X \) such that \( x > y \).

Hence, in a multiset comparison, an element can be replaced by any number (including 0) of smaller elements.

Alternatively, \( >_{\text{mul}} \) is the transitive closure of \( >_{\text{m}} \) where \( M >_{\text{m}} N \) iff there are \( Z, x \) and \( Y \) such that \( M = Z + [x] \), \( N = Z + Y \) and \( x > Y \).

Example: On \( \mathbb{N} \), we have: \[3] >_{\text{mul}} [2, 2, 2, 2] >_{\text{mul}} [2, 2, 2, 1, 1] >_{\text{mul}} [2, 2, 2, 1, 0, 0, 0] >_{\text{mul}} \ldots >_{\text{mul}} \emptyset \].

Proposition 1 • If \( > \) is a strict order, then \( >_{\text{mul}} \) is irreflexive.

• If \( > \) is transitive, then \( >_{\text{mul}} \) is transitive.

Lemma 3 \( >_{\text{mul}} \) is terminating order whenever \( > \) so is.

Proof. Dershowitz and Manna’s proof uses König’s lemma. Van Oostrom suggests that this property can be proved by slightly modifying Nash-Williams’ proof of Kruskal theorem. However, the simplest proof is due to Buchholz [10]:

First note that it is sufficient to prove the termination of \( >_{\text{m}} \).

To this end, we prove that, for all \( M, M \) terminates, by induction on the multiset cardinal of \( M \) (i.e. sum of multiplicities) (1). If \( M = \emptyset \), this is immediate. Otherwise, \( M = N + [x] \). By induction hypothesis (1), \( N \) terminates.

We then prove that, for all \( (x, N) \) such that \( N \) terminates, \( N + [x] \) terminates, by induction on \( (>_{\text{m}})(\leq_{\text{lex}}) \) (2). To this end, it suffices to prove that every reduct \( P \) of \( N + [x] \) terminates. There are two cases:

\[\text{\ldots}\]
\[ P = N' + [x] \text{ and } N >_m N'. \text{ Since } N \text{ terminates, } N' \text{ terminates. Therefore, by induction hypothesis (2), } P \text{ terminates.} \]

\[ P = N + Y \text{ and } x > Y. \text{ We then prove that, for all } Y < x, N + Y \text{ terminates, by induction on the multiset cardinal of } Y \text{ (3). If } Y = \emptyset, \text{ this is immediate. Otherwise, } Y = Y' + [y]. \text{ By induction hypothesis (3), } N + Y' \text{ terminates. Therefore, by induction hypothesis (2), } (N + Y') + [y] = N + Y \text{ terminates.} \]

Remark:

**Lemma 4** If \( M >_{\text{mul}} N \) and \( P \subseteq >_{(M - > (N))} \), then \( M >_{\text{mul}} N + P. \)

**Proof.** Assume that \( M = Z + X, N = Z + Y, X \neq \emptyset \) and \( Y \subseteq >_{(X)} \). The result holds since \( P \subseteq >_{(Z + X) - > (Z + Y)} = >_{(X)} - > (Z + Y) \subseteq >_{(X)} \).

\[ \square \]

2 The recursive path ordering (RPO)

Another reduction order is the recursive path ordering (RPO) introduced by Dershowitz in 1979 [4].

The idea is to extend a well-founded ordering on function symbols (prece-
dence) into a well-founded ordering on terms by comparing the head symbols and then the subterms recursively.

**Definition 2 (Recursive Path Ordering)** Given a quasi-order \( \succeq \) on \( \mathcal{F} \), let the multiset path ordering (MPO) \( >_{\text{rpo}} \) be the smallest relation on terms such that \( t >_{\text{rpo}} u \) if \( t = f \vec{t} \) and either:

1. \( t_i >_{\text{rpo}} u \) for some \( i; \)
2. \( u = g \vec{u}, f \succ \mathcal{F} g \text{ and } t >_{\text{rpo}} u_i \) for all \( i; \)
3. \( u = g \vec{u}, f \simeq_{\mathcal{F}} g, t >_{\text{rpo}} u_i \) for all \( i, \) and \( \vec{t}(>_{\text{rpo}})_{\text{mul}} \vec{u}. \)

The lexicographic path ordering (LPO) [6] is obtained by replacing \( (>_\text{rpo})_{\text{mul}} \) by \( (>_\text{rpo})_{\text{lex}}^n \) for some fixed \( n \) (e.g. the maximal arity of function symbols if it is finite). Finally, both can be combined into the recursive path ordering (RPO) by using a function status : \( \mathcal{F} \rightarrow \{\text{mul, lex}\} \) such that \( \text{status}(f) = \text{status}(g) \) whenever \( f \simeq g, \) and replacing \( (>_\text{rpo})_{\text{mul}} \) by \( (>_\text{rpo})_{\text{status}(f)} \).

**Theorem 2** \( >_{\text{rpo}} \) terminates whenever \( > \) terminates.

**Proof.** Dershowitz’ original proof works for finite signatures only. It consists in first proving that \( >_{\text{rpo}} \) is a transitive rewrite relation containing the superterm relation, and then proving that, on finite signatures, such a relation terminates since its inverse is a well-quasi-order (wqo) by Kruskal’s theorem [7, 8], which says that the tree embedding extension of a wqo on \( \mathcal{F} \) (the equality here) is a wqo on \( \mathcal{T(\mathcal{F}, \emptyset)} \).
Instead, we will follow Buchholz’ proof [2] which is not restricted to finite signatures. We prove that every term $t$ terminates by induction on $t$. If $t \in V$, this is immediate. Therefore, it remains to prove that, for all $f$ and terminating terms $t, ft'\, \text{terminates. We proceed by induction on } > \times_{\text{lex}} (\geq_{\text{rpo}})_{\mu}(1)$. To prove that $ft'$ terminates, it suffices to prove that, for all $u$ such that $ft' \geq_{\text{rpo}} u$, $u$ terminates. We prove this by yet another induction on $u$ (2):

1. $u$ terminates since $t_i \geq_{\text{rpo}} u$ and $t_i$ terminates.
2. By induction (2), $\bar{u}$ terminate. Therefore, by induction (1), $u$ terminates.
3. Idem.

References