A point on fixpoints

Frédéric Blanqui

INVENTEURS DU MONDE NUMÉRIQUE

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To whom to attribute the following “well-known” result?

Given a function f on a set of sets X, the iteration of f:

- a_0
- $a_{k+1} = f(a_k)$
- $a_\omega = \bigcup\{a_k | k < \omega\}$

converges to some fixpoint of f (*i.e.* $f(a_\omega) = a_\omega$)
if ... (some condition is satisfied)

Example 1: transitive closure of a relation R on some set A

If:

- $X = \mathcal{P}(A \times A)$
- $a_0 = R$
- $f(S) = R \circ S$ where $x\,(R \circ S)\,y$ if $(\exists z) xRz \wedge zSy$

Then:

- $x\,a_k\,y$ if one can go from x to y in k steps exactly
- a_ω is the transitive closure of R.

Example 2: Kleene first fixpoint theorem

Theorem: given a partial recursive functional $F(\zeta, x_1, \ldots, x_n)$ where ζ ranges over p.r. functions of n variables, there is a minimal p.r. function ζ s.t. $\zeta(x_1, \ldots, x_n) = F(\zeta, x_1, \ldots, x_n)$.

- X is the set of partial recursive functions of n variables
- a_0 is the function defined no where

To whom to attribute the following "well-known" result?

On a poset $(X, \leq)$ such that…(some condition), the transfinite iteration of some function $f : X \to X$:

- a_0
- $a_{k+1} = f(a_k)$
- $a_l = \mathrm{lub}\{a_k | k < l\}$ if l is a limit ordinal

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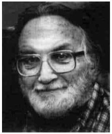
Let A be the set of all the iterates of f from a_0.

Let $FP(f)$ be the set of all the fixpoints of f.

Nobody because it is trivial?

If f is *extensive* ($x \leq f(x)$), then A is inductive (every chain[a] has a lub). By Tuckey's maximal principle, A has a maximal element a_k. Since f is extensive, $a_k \leq f(a_k)$. Since a_k is maximal, $f(a_k) \leq a_k$.

[a] A chain is a totally ordered subset.



Zorn's maximal principle (1935)

Any inductive set of sets has a maximal element wrt inclusion.

Tuckey's maximal principle (1940)

Any inductive poset has a maximal element.

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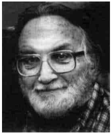

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Tuckey's maximal principle (1940)

Any inductive poset has a maximal element.

Zorn's and Tuckey's maximal principles are both equivalent to the Axiom of Choice introduced by Zermelo in 1904.

Cardinality argument?

"A cannot be bigger than X..."

but cardinal theory is based on the Axiom of Choice...

Searching for a proof not using the Axiom of Choice...

Here are useful references I started with:

- *Fixed point theorems and semantics: a folk tale*, J.-L. Lassez, V. L. Nguyen and E. A. Sonenberg, Information Processing Letters, 1982.
- *The origin of "Zorn's lemma"*, P. J. Campbell, Historia Mathematica, 1978.

but they were not quite sufficient...

Searching for the origins. . . 1909

The oldest reference I found is:[1]

Kettentheorie und Wohlordnung, Gerhard Hessenberg, Journal für die reine und angewandte Mathematik (Crelle), 1909.

Hessenberg (1909)

If X is a set of sets, $\leq$ inclusion and f extensive, then $FP(f) \neq \emptyset$.

I don't know if Hessenberg says anything about the iterates of f.

[1] I didn't check it by myself because it is in German, but this is explained in English on a mailing list on the history of mathematics in a mail written in 2000 by Felscher, who wrote a paper on this subject in German in 1962.

Searching for the origins. . . 1922

The first paper I found that speaks about the iterates of f is:

Une méthode d'élimination des nombres transfinis des raisonnements mathématiques, Casimir Kuratowski, Fundamenta Mathematicae, 1922.

Kuratowski (1922)

If X is a set of sets, $\leq$ is inclusion and f is extensive then:

- $A = N$, the smallest subset containing a_0 and closed by f and $\neq \emptyset$ lub's,
- $\mathrm{lub}(N) \in FP(f)$,
- if f is monotone then $\mathrm{lub}(N)$ is the smallest fixpoint of $f \geq a_0$.

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- if f is monotone then $\mathrm{lub}(N)$ is the smallest fixpoint of $f \geq a_0$.

1. $\mathrm{lub}(N) \leq f(\mathrm{lub}(N))$ since f is extensive
2. $\mathrm{lub}(N) \in N$ since N is closed by non-empty lub's
   $\Rightarrow f(\mathrm{lub}(N)) \in N$ since N is closed by f
   $\Rightarrow f(\mathrm{lub}(N)) \leq \mathrm{lub}(N)$ by definition of lub

Searching for the origins. . . 1927


Un théorème sur les fonctions d'ensembles, Bronislaw Knaster and Alfred Tarski, Annales de la Société Polonaise de Mathématiques, 1928 (one page note).

Knaster and Tarski (1927)

If X is a set of sets, $\leq$ inclusion and f monotone, then $FP(f) \neq \emptyset$.

(In particular, they use this to prove Cantor-Bernstein theorem. . .)

But nothing is said about the iterates of f. . .

Searching for the origins. . . 1939

In 1939, Tarski extended his result with Knaster to arbitrary complete lattices (every subset has a lub and a glb).

A Lattice-theoretical Fixpoint Theorem and its Applications, A. Tarski, Pacific Journal of Mathematics, 1955.[2]

Tarski (1939)

If X is a complete lattice and f is monotone, then $FP(f)$ is a complete lattice.

[2] In the same journal, Davis (a student of Tarski) proves the converse: a lattice is complete if every monotone map has a fixpoint.

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Tarski (1939)

If X is a complete lattice and f is monotone, then $FP(f)$ is a complete lattice.

Tarski also notes p. 305 that, if X is ω-complete (every *countable* subset has a lub and glb) and f is ω-continuous ($|X| \leq \omega \Rightarrow f(\bigvee X) = \bigvee f(X)$), then $a_\omega \in FP(f)$. This is also used by Kleene in his first recursion theorem in *Introduction to metamathematics*, North-Holland, 1952.

[2] In the same journal, Davis (a student of Tarski) proves the converse: a lattice is complete if every monotone map has a fixpoint.

Searching for the origins. . . 1939

In 1939, Bourbaki extended Hessenberg's theorem to posets:[3]

Bourbaki (1939)

If X is a non-empty strictly inductive poset (every *non-empty* chain has a lub) and f is extensive, then $\mathrm{lub}(N)$ is the *least* fixpoint of f.

This was later proved by other people: Kneser (1950), Szele (1950), Witt (1950), Vaughan (1952), Inagaki (1952), . . .

[3] But the proof was published in 1949 only.

Searching for the origins. . . 1959

In 1959, using Bourbaki's theorem, Abian and Brown extended Knaster and Tarski's result to strictly inductive posets having a pre-fixpoint of f:

A theorem on partially ordered sets with applications to fixed point theorems, Smbat Abian and Arthur B. Brown, Canadian Journal of Mathematics, 1961.

Abian and Brown (1959)

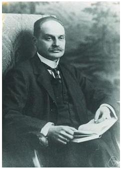
If X is a non-empty strictly inductive poset, f is monotone and $a_0 \leq f(a_0)$, then $FP(f) \neq \emptyset$.

Searching for the origins. . . 1963

In their book *Equivalents of the Axiom of Choice*, Herman Rubin and Jean E. Rubin prove the result by invoking Hartogs theorem:

Rubin and Rubin (1963)

If X is a set of sets, $\leq$ is inclusion and f is extensive, then there is k such that $a_{k+1} = a_k$.



Hartogs (1915)

For any set A, there is an ordinal k that cannot be injected into A.

After Hartogs theorem, $a|_k$ is not an injection. Therefore, there are $l_1 < l_2 < k$ such that $a_{l_1} = a_{l_2}$. Since f is extensive, $a_{l_1+1} = a_{l_1}$.

Searching for the origins... 1973

In 1973, using Bourbaki's theorem, Markowsky extended Tarski and Davis results to inductive posets:

Chain-complete posets and directed sets with applications, George Markowsky, Algebra Universalis, 1976.

Markowsky (1973)

If X is inductive and f is monotone, then $FP(f)$ is inductive.
X is inductive if every monotone map has a *least* fixpoint.

Searching for the origins. . . 1977

In 1977, Cousot and Cousot study also some properties of the iterates of f when f is monotone and X a complete lattice:

Constructive versions of Tarski's fixed point theorems, Patrick Cousot and Radhia Cousot, Pacific Journal of Mathematics, 1979

To summarize

- There are results:
  - on the existence of a fixpoint (Hessenberg, Knaster-Tarski, Bourbaki, Abian-Brown, Markowsky, . . .)
  - on the iterates of f (Kuratowski, Rubin-Rubin, Cousot-Cousot)

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- There are results:
  - on the existence of a fixpoint (Hessenberg, Knaster-Tarski, Bourbaki, Abian-Brown, Markowsky, ...)
  - on the iterates of f (Kuratowski, Rubin-Rubin, Cousot-Cousot)
- Two conditions are considered:
  - f is extensive (Hessenberg, Kuratowski, Bourbaki, Rubin-Rubin)
  - f is monotone (Knaster-Tarski, Abian-Brown, Markowsky, Cousot)

Relation between extensivity and monotony?

1. On a well ordered poset, a strictly monotone function is extensive (Bourbaki, 1953)

Assume f not extensive. Then $E = \{x \in X | f(x) < x\} \neq \emptyset$. Let ξ be the least element: $f(\xi) < \xi$. Since f is strictly monotone, $f(f(\xi)) < f(\xi)$. Thus $f(\xi) \in E$ and $\xi \leq f(\xi)$. Contradiction.

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   - $a_0 \leq f(a_0)$ and f monotone on $A \Rightarrow f$ extensive on A
   - f extensive on $A \Rightarrow a$ monotone $\Rightarrow f$ monotone on A

A condition generalizing both monotony and extensivity?

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Puntos fijos en conjuntos ordenados, Baltasar R. Salinas, Publicaciones del Seminario Matematico Garcia de Galdeano, Facultad de Ciencas de Zaragoza, 1969

Salinas (1969)

On a poset X every $\neq \emptyset$ well-ordered subset of which has a lub, a function f has a fixpoint if:

(P1) $a_0 \leq f(a_0)$,

(P2) $x \leq f(x) \leq y \;\Rightarrow\; f(x) \leq f(y)$.

Moreover, under AC, there is k such that $a_k \in FP(f)$.

Note that (P2) is satisfied whenever f is monotone or extensive

He also proved the converse: every $\neq \emptyset$ well-ordered subset has a lub if every function satisfying (P1) and (P2) has a fixpoint.

My two cents

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   - f satisfies (P1) and (P2) $\Rightarrow a$ monotone (by transfinite induction)

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1. We can prove that $FP(f) \cap A \neq \emptyset$ without using AC:
   - a monotone $\Rightarrow FP(f) \cap A \neq \emptyset$ (by Hartogs theorem)
   - f satisfies (P1) and (P2) $\Rightarrow a$ monotone (by transfinite induction)
2. Using a result of Abian-Brown (1959), we can slightly weaken:

   $$\text{(P2) } x \leq f(x) \leq y \;\Rightarrow\; f(x) \leq f(y)$$

   by:

   $$\text{(P2') } x < f(x) \leq y \wedge]x, f(x)[= \emptyset \;\Rightarrow\; f(x) \leq f(y)$$

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Conclusion: to whom to attribute the well-known result?

- If f is extensive and:
  - X is a set of sets: Kuratowski (1922)
  - X is a strictly inductive poset: Bourbaki (1939) for $\mathrm{lub}(N) \in FP(f)$ and Kuratowski (1922) for $N = A$
- If f is monotone, $a_0 \leq f(a_0)$ and:
  - X is an ω-complete lattice and f is ω-continuous: Tarski (1939)
  - X is a strictly inductive poset: Abian and Brown (1959) for $\mathrm{lub}(N) \in FP(f)$ and Kuratowski (1922) for $N = A$
- If f satisfies (P2), $a_0 \leq f(a_0)$ and X is a strictly inductive poset: Salinas (1969)

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Thank you!

Abian and Brown (1959)

Definition: A set $C \subseteq X$ is an a_0-*chain* if:

- C is well ordered
- $\mathrm{glb}(C) = a_0 C$ has a_0 as least element
- C is closed by non-empty lub's
- if $z \in C - \{\mathrm{lub}(C)\}$ then:
  - $f(z) \in C$
  - $z < f(z)$
  - $]z, f(z)[\cap C = \emptyset$

Let $W = \{\mathrm{lub}(C) | C \text{ is an } a_0\text{-chain}\}$.

Abian and Brown (1959)

Theorem: For any poset $(X, \leq)$, $a_0 \in X$ and function $f : X \to X$:

- W is well ordered
- W has a_0 as least element
- if W has a $\mathrm{lub}\ \xi$, then W is an a_0-chain with ξ has greatest element and $\xi \not< f(\xi)$